CAP PRIZE EXAMINATION 2008

Compiled by the Physics and Astronomy Department, University of Victoria

Problem 1. Mechanics: Spinning hockey puck
Determine the amount of time required for a hockey puck spinning on its flat side to come to a complete stop. The radius of the puck is R, its mass is m, the initial angular velocity is ω, and the friction coefficient between the puck and the ice is μ.

Problem 2. Mechanics: Period of oscillation in the quartic potential
A point-like particle moves in the field of a one-dimensional quartic oscillator that has the potential energy $V(x) = ax^4 + bx^3 + cx^2 + dx + e$. The parameters $a,\ b,\ c,\ d$, and e are arbitrary, up to the condition that the potential allows for two separate regions of finite motion corresponding to the same total energy E, as shown in Figure 1.

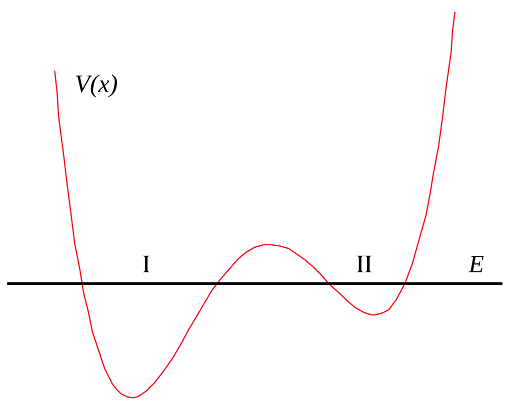

Figure 1:

Prove that for the same energy E, the periods of oscillation in region I and II are equal,

$$T_{\mathrm{I}} = T_{\mathrm{II}} \ .$$

Problem 3. E&M: Motion in external fields
An alloy of lithium and sodium is analyzed for its elemental composition using the following method. It is vaporized and partially ionized by heating to a temperature of 2000K. Then the ionized fraction is accelerated in a uniform longitudinal electric field to energy $E \gg kT$ to form a beam that is then sent through an "analyzing" transverse uniform magnetic field. The ions are counted by a position-sensitive detector located in the plane perpendicular to the beam direction that registers the number of ions as a function of their deviation from

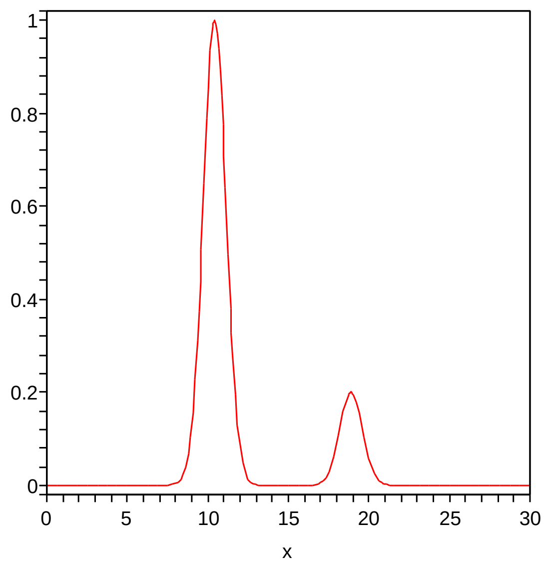

Figure 2:

a straight line trajectory. The resulting image (in arbitrary units) is shown in Figure 2.
a) Find the relative concentration of Na and Li in the original alloy. The ionization energy for sodium is $I_{\mathrm{Na}} = 5.139$ eV, and for lithium is $I_{\mathrm{Li}} = 5.392$ eV.
b) From Figure 2, determine the ratio of masses for the sodium and lithium atoms, if it is known that $x = 0$ corresponds to the beam trajectory with zero magnetic field. Assume that the the deflection angles in the magnetic field are small.

Problem 4. E&M: Tangential sphere capacitance problem

Given that the capacitance of a conductor with charge Q is defined as Q/V, where V is the potential on the surface (relative to 0 at infinity), find the capacitance C of a conductor composed of two tangent spheres of radius R, as shown in Figure 3.

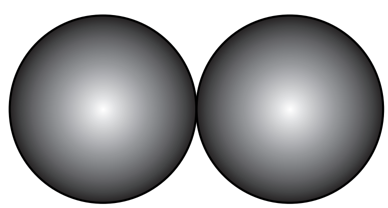

Figure 3:

Problem 5. Condensed matter: Phonon emission

Neutrons propagating in matter have the quadratic dispersion relation, $E_n(k) = (\hbar k)^2/(2m_n)$. Under certain conditions, they can lose energy by emitting phonons that have a linear dispersion relation, $E_{ph}(k) = c_s(\hbar k)$, where c_s is the speed of sound. Find the condition on neutron energy that allows for energy loss via phonon emission. Give a numerical estimate of the critical energy using typical values of c_s in solids.

Problem 6. Statistical mechanics: Bose-Einstein condensation in an external field

Consider an ideal gas of non-relativistic particles with Bose-Einstein statistics (spin=0) in an external field acting along the z-direction,

$$V(x, y, z) = \frac{1}{2} m\omega^2 z^2.$$

Treat the particle motion in this potential classically. Find the critical temperature T_0 below which the Bose-Einstein condensation phenomenon will take place. The mass of the particles is m, the total number of particles is N, and the motion in the xy-directions is limited by the area A (*i.e.* the number density of particles per unit of xy-surface area is $\sigma = N/A$). Sketch the distribution of particles along the z-coordinate for temperatures $T < T_0$.

Hints: The Bose-Einstein distribution is given by

$$dN = \frac{d^3r d^3p}{(2\pi\hbar)^3} \frac{1}{\exp\left(\frac{\epsilon(\mathbf{r},\mathbf{p})-\mu}{kT}\right) - 1},$$

where $\epsilon(\mathbf{r}, \mathbf{p})$ is the one-particle energy, and μ is the chemical potential that vanishes at $T = T_0$. A useful integral: $\int_0^\infty x dx/(e^x - 1) = \pi^2/6$.

Problem 7. Statistical mechanics: Diffusional broadening of the beam

A well-collimated [idealized] cold beam of particles with mass M, longitudinal velocity $v_\parallel = v_0$, and transverse velocity of particles $v_\perp = 0$, enters a chamber with gas kept at temperature T. The collisions between the gas and beam particles are elastic and isotropic, and have the cross section σ_0. Estimate the characteristic transverse velocity of beam particles $(v_\perp^2)^{1/2}$ after a travel distance L in the gas for the two separate cases:

a) Estimate $v_\perp(L)$ if the particles in the gas are nonrelativistic, with mass m ($m \ll M$), and number density n. Assume $n\sigma_0 L \gg 1$ and neglect possible changes in the longitudinal velocity of beam particles. Sketch the dependence $|v_\perp(L)|$.

b) Repeat part a) for a gas of photons of temperature T.

Problem 8. Quantum theory: Hydrogen molecular ion binding energy using Bohr quantization

The Hydrogen molecular ion H_2^+ is the simplest diatomic molecule consisting of two protons and one electron. Use the Bohr-de Broglie orbit quantization condition – that the length of the electron orbit is equal to an integer number of de Broglie wavelengths – to find the total binding energy of this system. To achieve this, assume a circular electron trajectory in the plane equidistant from both protons (Figure 4), and calculate the total ground state energy E as a function of proton separation R_p; the minimum value of $E(R_p)$ gives the binding energy. How does the your result compare with the full quantum mechanical answer of $E = -16.4$eV? (Note that H_2^+ is 2.8eV more deeply bound than the system of a hydrogen atom plus a free proton). If you find a disagreement, is there a physical reason for it? Can the kinetic energy of the protons be neglected in this calculation?

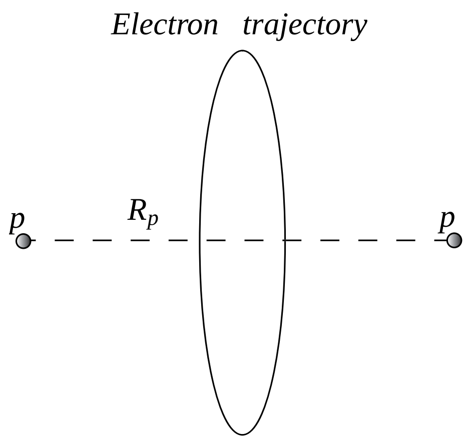

Figure 4:

Hints: Minimization of $E(R_p)$ needs to be done numerically, and so an estimate with $\sim 10 - 15\%$ accuracy will suffice. For convenience, measure all energies in units of the hydrogen binding energy, $E_b = 13.6\text{eV}$, and all distances in units of the Bohr radius, $a_B = 5.3 \times 10^{-9}\text{cm}$.

Problem 9. Quantum mechanics: Tunneling through a wall

Consider the quantum mechanical motion of a particle with mass m confined by the one-dimensional potential $V(x)$. The potential (shown in Figure 5) is represented by a thin (delta-functional) barrier that partitions the infinite potential square well into two equal-size "compartments" of length a:

$$V(x) = \begin{cases} +\infty & \text{for } \ |x| > a, \\ g\delta(x) & \text{for } \ |x| < a, \quad \text{where } \ g > 0, \end{cases}$$

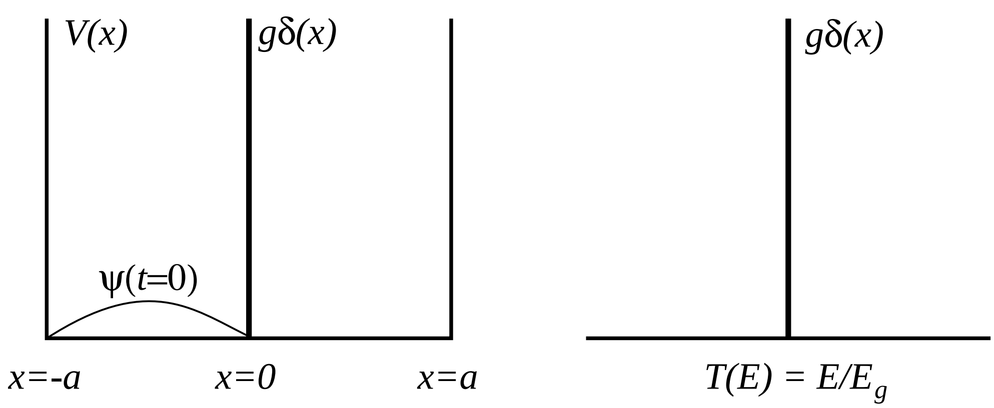

Figure 5:

a) It is known that the transmission coefficient (the ratio of transmitted to incident particle flux) through an isolated delta-functional barrier is small and given by $T(E) =$

E/E_g for energies of incident particles on the order of $\hbar^2/(ma^2)$. Find E_g as a function of the strength of the delta-functional barrier g.

b) Find approximately, in an expansion in the small parameter $\hbar^2/(ma^2E_g)$, the first eigenenergies and the corresponding wave functions for the two lowest energy quantum states in the potential $V(x)$.

c) Calculate the evolution in time of a state initially concentrated entirely in the left compartment, and described by the wavefunction

$$\psi(x, t = 0) = \begin{cases} -\sqrt{\frac{2}{a}}\sin(\pi x/a) & \text{for} \quad -a < x < 0, \\ 0 \ \text{elsewhere} \end{cases} .$$

Determine the time t_0 needed to "tunnel through the wall" given by the condition $\int_0^a |\psi(x, t = t_0)|^2 dx \sim O(1)$.

Problem 10. Equivalence principle

A piece of rock on the surface of a planet contains some amount of the element X with a lifetime τ_0 for radiative decay into another element Y. A powerful explosion breaks the rock into several pieces. One piece (piece A) stays on the surface near the explosion point, while the other piece (piece B) is sent vertically upward, to be returned to the surface by the gravitational force of the planet after time T, as measured by clocks at the surface. Upon return, the ratios of N_X/N_Y in pieces A and B are compared (as they may get affected by relativistic and gravitational effects). Assuming that before the explosion X and Y are distributed uniformly inside the rock, give a qualitative comparison of N_X/N_Y in the two pieces A and B after the return of piece B and justify your answer. Assuming the velocity of piece B after the explosion is nonrelativistic, there is a uniform gravitational acceleration g, and the "time of flight" T is much shorter than τ_0, calculate the difference in the percentage of decayed atoms of element X in samples A and B.

Physical constants used in the exam problems:

Speed of light $c = 3 \times 10^{10}$cm/s
Boltzmann constant $k = (1\text{eV})/(11600\text{K}) = 1.38 \times 10^{-23}$J/K
Planck constant $\hbar = 2 \times 10^{-5}$ eV cm/$c = 1.1 \times 10^{-34}$ J s
Electron mass $m_e = 5.1 \times 10^5$ eV/$c^2 = 9.1 \times 10^{-31}$ kg
Neutron mass $m_n = 0.94 \times 10^9$ eV/$c^2 = 1.7 \times 10^{-27}$ kg
Proton mass $m_p = 0.94 \times 10^9$ eV/$c^2 = 1.7 \times 10^{-27}$ kg
Fine-structure constant $\alpha = e^2/(\hbar c) = 1/137$ (In SI units $\alpha = e^2/(4\pi\epsilon_0\hbar c)$)
Bohr radius $a_B = \hbar/(\alpha m_e c) = 5.3 \times 10^{-9}$cm
Hydrogen binding energy $E_b = \hbar^2/(2m_e a_B^2) = \frac{1}{2}\alpha^2 m_e c^2 = 13.6$ eV