2002 CAP University Prize Examination

Wednesday, February 6

From 2:00 to 5:00 p.m.

The use of calculators is permitted.

Each question is to be answered in a separate book. The number of the question, the name of the candidate, and the name of the university/department must be indicated clearly on the first page.

The candidate should attempt as many questions as possible, in whole or in part. It is not expected that all questions will be completed.

Each question holds the same value.

1) Optics/Astronomy

Telescopes are afocal instruments for the observation of the heavenly objects. They are both composed of an objective and an eyepiece. The objective may be a lens (refracting telescope) or a mirror (reflecting telescope).

1. The objective of a refracting telescope has a focal distance f_1. Its eyepiece has a focal distance f_2. Schematize the principle of operation of the instrument by ray tracing (for a tilted ray impinging at an angle α with respect to the axis). Define and calculate the angular magnification G_a of the telescope.
Give the numerical value of G_a for f_1 = 16 m, f_2 = 4 cm.

2. Explain the advantages of a catadioptric system compared to a dioptric system.

3. In a Cassegrain telescope, the main mirror is concave and parabolic. It is associated with a small convex mirror. Rays reflected by this second mirror converge through an opening in the primary mirror. Make a diagram of the objective thus made up and explain its advantages.

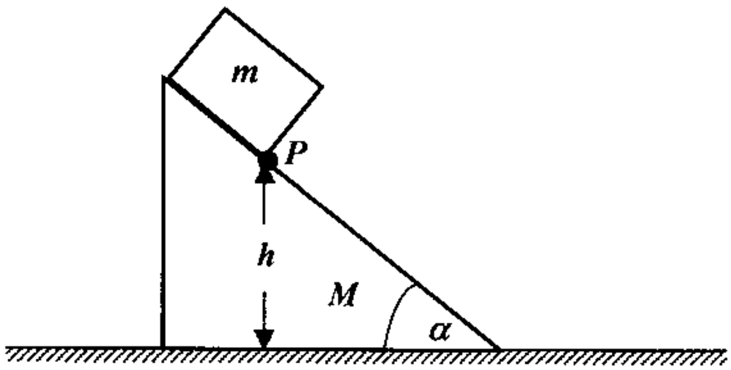
4. The principal mirror of the telescope of the Palomar Mountain has a diameter of 5,08 m (200″). Taking into account diffraction, give the limit of the angular resolution θ of two sources emitting in the visible range (i.e. for the average wavelength λ = 500 nm) ?

5. Same question but for the radio telescope of Arecibo (Puerto Rico) having a principal mirror of diameter of 305 m and operating at a wavelength λ' = 80 m.

2) Classical Mechanics

A block of mass *m* rests on a wedge of mass *M* which, in turn, rests on a horizontal table, as shown in the Figure. All surfaces are frictionless.

If the system starts at rest with point *P* of the block at the distance *h* above the table, find the velocities of the wedge and the block (with respect to the table) at the instant point *P* touches the table.



3) Thermodynamics

Air (an ideal gas) in an adiabatic container of volume V is initially at P_0 and T_0. The air in the container is to be cooled at constant pressure by introducing air at P_0 and $T_1 < T_0$, at a constant mass flow rate of $\dot{m}_{in}$. The pressure in the container is held constant by adjusting the mass flow rate, $\dot{m}_{out}$, of air at P_0, whose temperature is that of uniform mixture of the gas in the container.

a) Prove that the internal energy, U = mu, of the gas in the container is constant.

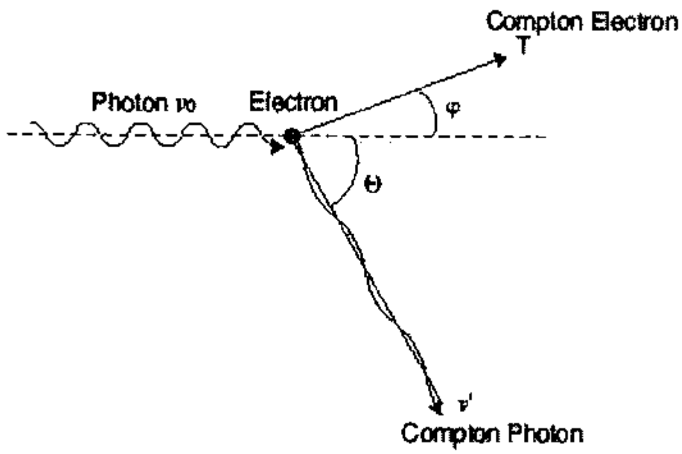
b) Assuming that the container is rigid, and neither absorbs no emits heat, give the equations for the first law of thermodynamics in uniform flow, and for continuity, for this situation.

c) Using the equations presented in (b), derive the differential equation for the temperature, T (t), in the container, as a function of time.

4) Nuclear Physics (Compton Effect)

Consider a free electron at rest, and an incident photon whose energy is of the order of the rest energy of the electron. The problem geometry is illustrated on Figure 1.

Figure 1: Trajectories



Using relativistic conservation laws, show that:

$$\frac{\nu'}{\nu_0} = \frac{1}{1+\alpha(1-\cos\Theta)}$$

1. $T = h\nu_0 \dfrac{\alpha(1-\cos\Theta)}{1+\alpha(1-\cos\Theta)}$

2. $\cot\varphi = (1+\alpha)\tan\dfrac{\Theta}{2}$

where:

- $\alpha = \dfrac{h\nu_0}{m_0 c^2}$

- T is the kinetic energy of the electron after the collision
- Θ is the angle made by the electron after the collision
- φ is the angle made by the photon after the collision
- m_0 is the electron rest mass

5) Relativity

We study the collision of two protons P_1 et P_2, each having a mass M_p.

1. The proton P_2 is at rest in the laboratory system (S). E and E' designate the total energies of the two protons in the laboratory system (S) and in the center of mass system (S'), respectively. Show that:

$$\frac{E'}{E} = \frac{2M_p c^2}{E'}$$

2. The collision yields a π^+-meson, a proton P and a neutron N:

$$P_1 + P_2 \rightarrow \pi^+ + P + N.$$

Assume that after the collision, all particles are at rest with reference to (S').

a) Calculate the velocity of each proton P_1 and P_2

i) in the center of mass system (S');

ii) in the laboratory system (S).

b) In those conditions, what is the kinetic energy of P_1 in the laboratory system (S)?

Numerical data:

Proton self-energy $M_P c^2 = 938$ MeV

Neutron self-energy $M_N c^2 = 940$ MeV

π-meson self-energy $M_\pi c^2 = 140$ MeV.

6) Quantum Mechanics

A harmonic oscillator is in the initial state at t = o

$$\Psi(x,o) = \psi_1(x) + 7\,\psi_2(x)$$

a) Find the wavefunction at a later time.

b) Evaluate ΔE (t).

c) Evaluate Δx (t).

d) Is the product ΔE (t) Δx (t) an uncertainty relation?

Useful relationships:

$$<n/x/n-1> = \left(\frac{n}{2\alpha}\right)^{1/2}$$

$$<n/x/n+1> = \left(\frac{n+1}{2\alpha}\right)^{1/2}$$

$$<n/x^2/n> = \left(\frac{2n+1}{2\alpha}\right)$$

where $\alpha = m\omega/\hbar$.

7) Solid State (Conduction in metals)

(a) Assume that the conduction electrons in a metal , of number density n, are free and determine the velocity v_F of the most energetic ones (those with the Fermi energy).

(b) According to Drude's model the equation of motion for the velocity $\mathbf{v} = \mathbf{p}/m$ of an average conduction electron in the metal is

$$d\mathbf{p}/dt + \mathbf{p}/\tau = \mathbf{F}$$

where **F** is the force on the electronic charge (-q) due to the presence of external fields (the Lorentz force), and τ is the collision time (determined by the electron scattering processes in the solid) . Determine the drift velocity of the conduction electrons in the presence of a uniform, constant electric field **E**, and derive the carrier mobility μ and the conductivity σ.

(c) Determine the mean free path (mfp) of a conduction electron in a copper sample knowing that its electrical resistivity is $1.67 \times 10^{-8}\ \Omega$–m. Compare with the experimental value of 42 nm. Clearly show your work.

(d) Estimate the mfp at 300 °C. Write down your hypothesis.

(e) Describe an experiment that will allow the measurement of the mfp.

Data:

Cu : crystalline structure fcc (lattice constant 0,361 nm); atomic configuration $[Ar]3d^{10}4s^{1}$
Elementary charge $q = 1{,}602 \times 10^{-19}$ C
Planck's constant: $\hbar = 1{,}054 \times 10^{-34}$ J – s
Electron mass : $9{,}11 \times 10^{-31}$ kg

8) Solid State (Plasmons)

This question is related to plasmons, collective excitations of the electron gas in a solid. Consider a semi-infinite plasma with relative permittivity $\kappa'(\omega)$ in a solid which occupies the half-space on the positive side of the plane z = 0. A solution of the Laplacian $\nabla^2 \phi = 0$ in the plasma is given by

$$\phi_p \; (x, z) = A \cos kx \; e^{-kz}$$

a) By using the appropriate boundary condition, show that the potential in the vacuum (z < 0) is

$$\phi_v = A \cos kx \; e^{kz}$$

b) Using another boundary condition at the interface, show that here

$$\kappa'(\omega) = -1$$

and, therefore, $\omega_s^2 = 1/2\,\omega_p^2$

(Note: the "plasma frequency" is defined by $\kappa'(\omega) \equiv 1 - \frac{\omega_p^2}{\omega^2}$; ω_s is the angular frequency of a surface plasmon).

9) Crystallography

The diffraction of X rays by a crystal lattice is described by Bragg's law

$$2\,d_{hkl}\ \sin\theta = n\lambda \tag{1}$$

where λ is the wavelength of the X ray beam, incident at angle θ on a set of lattice planes with Miller indices (hkl). The spacing between neighbouring planes is d_{hkl}, and n is an integer (the order of diffraction). Note that (1) can also be written

$$\left.\begin{array}{l} 2d_{nh\ nk\ nl}\sin\theta = \lambda \\ \text{or} \\ 2\,d_{HKL}\ \sin\theta = \lambda, \end{array}\right\} \tag{2}$$

where HKL may have an integral factor, n, in common. Lattice spacings, d_{HKL}, obtained from a Debye-Scherrer (powder) diffraction recording are (all in Å units):

3.24; 3.13; 2.81; 2.21; 1.98; 1.81; 1.69; 1.62; 1.56; 1.40

a) Show that the powder is comprised of a mixture of two distinct, cubic crystalline species, one face-centred-cubic (F), the other primitive (P). List the HKL values corresponding to the observed diffractions for each of the two species.

b) Determine the lattice parameters, a_0, for each of the two species (Note: an average value of a_0 will suffice in each case).

10) Electromagnetism

We recall Maxwell's equations in differential form:

$$\vec{\nabla} \bullet \vec{D} = \rho$$
$$\vec{\nabla} \bullet \vec{B} = 0$$
$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Let us consider the electric field of an electromagnetic wave in an isotropic, homogeneous dielectric medium in the following form:

$$\vec{E} = E_0 \cos(u\,x)\cos(\beta z - \omega t)\,\vec{y}$$

1. Calculate the magnetic field $\vec{H}$ of the wave.

2. a) What is the direction of the wave vector?
   b) Is the wave plane? Transverse?
   c) Calculate the phase velocity in the direction of propagation.

3. a) Calculate the Poynting vector associated with the wave and its time-average value.
   b) What is the direction of propagation of the energy?

4. What is the state of polarization of the wave?

5. How can one create such a wave by using uniform plane waves?