2000 CAP University Prize Examination
Thursday, February 10, 2000, 2:00 - 5:00 p.m.

Instructions:

1. Calculators are allowed.

2. You are most obviously not expected to complete all questions. You should attempt to answer carefully as many of them as possible, in whole or in part.

3. Answer each question in a different booklet, with the question number, your name and the name of your University clearly written on the first page.

4. The value attributed to each question partly takes into account the fact that the various questions are of very unequal lengths.

Question 1. (10 marks)

A balloon filled with helium is fixed to the floor of a plane with a string. As long as the plane is at rest, the balloon stretches the string along the vertical. The plane starts off with a constant horizontal acceleration **a**. Determine the angle between the string and the vertical, once the balloon has reached its position of equilibrium. Present clearly all the steps leading to your answer and draw a diagram showing the various relevant vectorial quantities.

Question 2. (15 marks)

Show that one cannot trap an ion with an electrostatic field in a region of empty space that contains no charges. Could it be done with a magnetostatic field?

Question 3. (15 marks)

(a) Consider a chain of N coherent oscillators separated by a distance a, each of which emits a spherical wave of wavelength λ. Assume that there is a phase shift ϵ between each oscillator and the next one. Show that the principal maximum of the Fraunhofer diffraction pattern is in the direction

$$\theta_0 = \arcsin\left(\frac{\epsilon\lambda}{2\pi a}\right) .$$

(b) A plane wave strikes the plane interface between two media at an angle θ_i with the normal. Assume that there is a chain of atoms along the intersection line of the surface and the plane of incidence, and consider them as a chain of coherent sources excited by the incident beam. Show that only a principal maximum is re-emitted into the incident medium when $\lambda >> a$, and that it is in the direction $\theta_0 = \theta_i$. This corresponds to the reflection law.

(c) Prove by a similar argument the refraction law of Snell-Descartes.

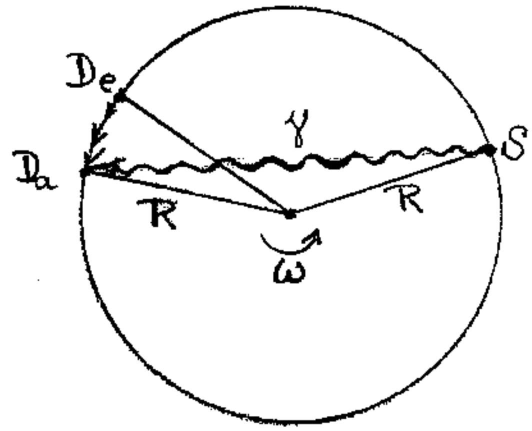
Question 4. (15 marks)

A plane mirror moves in the direction of its normal with a constant velocity v with respect to an inertial frame of reference. A beam of light of frequency ν is reflected by this mirror. It strikes it at an angle θ with the normal. What will be the reflection angle ϕ and the frequency of the reflected wave?

Question 5. (15 marks)

(a) A particle with 4-momentum $\mathbf{p}$ is examined by an observer with 4-velocity $\mathbf{U}$. Show that the energy he measures is given by $E = -\mathbf{p} \cdot \mathbf{U}$, with the signature $(-, +, +, +)$.

(b) The Mössbauer effect is used to realize the following high-precision redshift experiment. A source of γ rays of frequency ν and a detector are attached to the rim of a centrifuge rotating with a constant angular velocity ω. The angle between the emitter and the detector is measured to be φ in a frame of reference where $\omega = 0$. On the accompanying diagram, S, and D_e are the points of the laboratory where the source and the detector are, respectively, when emission of the γ ray takes place, while the detector reaches the point D_a at the time of absorption. Compute the redshift that is measured.



Question 6. (10 marks)

During an air trip, you read a paper on positronium, which is the atom you get by replacing the proton in a hydrogen atom by a positron . You would like to compare the binding energy and the radius of the positronium with those of the hydrogen atom, but unfortunately you don't know by heart the formulas for these quantities. You only remember that $M_p = 938$ MeV and $M_e = 0.5$ MeV. Still, you manage to compare the quantities you are interested in. How do you do it?

Question 7. The Thomas-Reiche-Kuhn sum rule (15 marks)

A particle of mass m moves in a potential $V(\mathbf{r})$ that is not velocity-dependant. Prove the relation

$$\sum_n \frac{2m \mid < n|x|0 > |^2}{\hbar^2} (E_n - E_0) = 1 \ ,$$

where the sum is over the complete basis of eigenvectors $|n>$, $(T+V)|n>= E_n|n>$.

Do you know (or can you guess) what the usefulness of this expression is?

Question 8. Localised phonons (30 marks)

Consider a periodic one-dimensional lattice (i.e. a chain) of N identical atoms, of mass m, at distance a from one another (at equilibrium). Interactions are with nearest-neighbours only and are characterised by a harmonic force constant K. The longitudinal displacement of an arbitrary atom n is described by the variable u_n.

(a) Give the equation of motion for atom n; derive the dispersion relation $\omega(k)$, expressed in terms of the speed of sound c_s. What is the maximum vibrational frequency ω_M?

(b) One of the atoms is replaced by an impurity of mass $M < m$; it is assumed that the force constants remain unchanged (i.e., $= K$). Obtain the equation of motion for (i) the impurity (displacement u_0) and (ii) an arbitrary atom (displacement u_n).

(c) We consider solutions for the displacements of atoms expressed in the form of a decaying wave, $u_n = u_0 e^{-\alpha|x|} e^{i(\Omega t - kx)}$, where $x = \pm na$, at the edge of the Brillouin zone, i.e. at $k = \pi/a$. Prove that a solution exists if $e^{\alpha a} = (2m - M)/M$. What is this solution? Is there another solution? What does α correspond to? Verify that the expected solution is recovered when $M = m$. How do Ω and ω compare?

(d) Evaluate Ω in the case $M = m/2$; compare to ω_M. Make a sketch of the instantaneous position of atoms as a function of x, starting at $x = 0$.

Question 9. Motion of an electric charge in the field of a magnetic monopole (30 marks)

The magnetic monopole, if such a thing exists, is the magnetic analogue of a point electric charge. Thus, a magnetic monopole of magnetic charge g fixed at the origin creates a magnetic field $\mathbf{B} = g\mathbf{r}/r^3$. One is immedistely faced with a problem when attempting to express this field as the curl of a vector potential $\mathbf{A}$. Indeed, just as in the electric case, $4\pi g = \int_S \mathbf{B} \cdot d\mathbf{S}$ for any surface S enclosing the origin. Now, this integral vanishes by the divergence theorem if the function $\mathbf{A}$ is a regular function, since, then, $\nabla \cdot \mathbf{B} = \nabla \cdot \nabla \times \mathbf{A} \equiv 0$. One can easily overcome this difficulty, though, by defining two regions R_a and R_b and by introducing two vector potentials, $\mathbf{A}_a$ over R_a and $\mathbf{A}_b$ over R_b. These regions, can be defined as follows:

$$R_a: \qquad 0 \leq \theta < \pi/2 + \delta \ , \qquad r > 0 \ , \qquad 0 \leq \varphi < 2\pi \ ,$$

and

$$R_b: \qquad \pi/2 - \delta < \theta \leq \pi \ , \qquad r > 0 \ , \qquad 0 \leq \varphi < 2\pi \ ,$$

with $0 < \delta \leq \pi/2$. The whole of space can be covered with them and they overlap for $\pi/2 - \delta < \theta < \pi/2 + \delta$. We will use the potentials

$$\mathbf{A_a} = g\frac{(1-\cos\theta)}{r\sin\theta}\mathbf{e}_\varphi \qquad \text{and} \qquad \mathbf{A_b} = -g\frac{(1+\cos\theta)}{r\sin\theta}\mathbf{e}_\varphi \ ,$$

here expressed in spherical coordinates, with the notation $\mathbf{A} = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_\varphi\mathbf{e}_\varphi$, and the usual set of orthonormal unit vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi\}$.

(a) Check that both of these potentials would be singular were they defined over the whole of space, but that such is not the case in the regions where they are defined.

(b) A particle of mass m and electric charge e moves in region R_a under the influence of the potential $\mathbf{A}_a$. Its equations of motion can be obtained from the Lagrangian

$$L_a = \frac{1}{2}m\dot{\mathbf{r}}^2 + \frac{e}{c}\dot{\mathbf{r}} \cdot \mathbf{A}_a \ .$$

Check that the Euler-Lagrange equations do indeed describe the motion of the charged particle under the Lorentz force exerted by the monopole.
N.B. In spherical coordinatees, $\mathbf{r} = r\ \mathbf{e}_r$ and $\dot{\mathbf{r}} = \dot{r}\ \mathbf{e}_r + r\dot{\theta}\ \mathbf{e}_\theta + r\sin\theta\dot{\varphi}\ \mathbf{e}_\varphi$. Since, moreover,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\mathbf{e}_\varphi \ ,$$

the following relations hold between the spherical components of the generalized force and those of $\mathbf{F}$:

$$Q_r = F_r \ , \quad Q_\theta = rF_\theta \ , \quad Q_\varphi = r\sin\theta F_\varphi \ .$$

c) In a similar fashion, the motion of the charged particle in R_b can be obtained from

$$L_b = \frac{1}{2}m\dot{\mathbf{r}}^2 + \frac{e}{c}\dot{\mathbf{r}}\cdot\mathbf{A}_b \ .$$

Show that L_a and L_b can be related by a simple gauge transformation in the region where R_a and R_b overlap. What does this imply for the motion of the charged particle?

d) Show directly, by using the equations of motion, that the vector

$$\mathbf{J} = \mathbf{r}\times m\dot{\mathbf{r}} - \frac{ge}{c}\mathbf{e}_r$$

is conserved. Can you guess what physical quantity is represented by the second member on the right-hand side? Can you prove that you have guessed well?

N. B. Useful relation: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$

(e) Show that the particle is moving on a cone. What are its apex, axis and semi-vertical angle?

Question 10. Degenerate Fermi gase and Chandrasekhar's limiting mass (30 marks)

Consider a completely degenerate Fermi gas.

(a) Show that the Helmoltz free energy is given by

$$F = \frac{4\pi V g}{h^3}\int_0^{p_F} e(p)p^2 dp \ .$$

(b) Show that in the non-relativistic limit, the pressure has the following density dependance

$$P \propto \left(\frac{N}{V}\right)^{5/3} \ .$$

(c) Show that in the ultra-relativistic limit, one rather has

$$P \propto \left(\frac{N}{V}\right)^{4/3} \ .$$

(d) The mechanical equilibrium of a spherical star is specified by the equations

$$\frac{dP}{dr} = -\rho\frac{Gm}{r^2} \ ,$$
$$\frac{dm}{dr} = 4\pi r^2\rho \ ,$$

where P is the pressure, ρ the matter density, r the radial coordinate and $m(r)$ the mass inside a sphere of volume $4\pi r^3/3$. Show that there is a relation of the form $M \propto R^{-3}$ between the mass M and the radius R of degenerate stars, when the fermions are non-relativistic.

(e) Show that when the fermions become ultra-relativistic, there exists a limiting mass above which a degenerate star cannot exist.

Hints:

$$F(N,V,T) = -kT \ln Z(N,V,T) \ , \qquad P = -\left.\frac{\partial F}{\partial V}\right|_{N,T}$$

$$\ln Z = \alpha N + \int_0^\infty \ln(1 + e^{-\alpha-\beta e(p)}) \frac{Vg}{h^3} 4\pi p^2 dp \ , \qquad N = \int_0^\infty \frac{1}{(e^{\alpha+\beta e(p)} + 1)} \frac{Vg}{h^3} 4\pi p^2 dp$$

For parts (d) and (e), first obtain the relation $3\int PdV = \int \frac{Gm}{r} dm$, find out the physical meaning of both sides of this equation and from there argue with averaged quantities.

Question 11. Ekman's spiral (30 marks)

We will consider in this problem **local** variations of the wind in magnitude and direction from the ground level up to altitudes of approximately $500m$. One can thus introduce a local Cartesian frame $Oxyz$, with axes Ox and Oy pointing southward and eastward, respectively.

On a large distance scale ($> 50km$) and at large altitudes, the horizontal motion of masses of air results from an equilibrium between the Coriolis force and the horizontal gradient of pressure ("geostrophic" equilibrium), and for a dominating wind blowing southward one has:

$$fU = -\frac{1}{\rho}\frac{dp}{dy} \ , \tag{1}$$

where U is the wind velocity along Ox, ρ the air density and p the horizontal pressure; finally, $f = 2\Omega\sin(\phi)$ is the Coriolis parameter, ϕ being the latitude and Ω the angular velocity of the Earth.

On the smaller distance scales of interest here, the two important forces are the Coriolis force and the viscous forces. We will assume we are at latitude $\phi = 45°$ North. The velocity of wind at altitude z, $u(z)\mathbf{i} + \mathrm{v}(z)\mathbf{j}$, is then approximately determined by the set of equations

$$-fv = \nu d^2u/dz^2 \ , \tag{2}$$

$$fu = \nu d^2v/dz^2 + fU \ . \tag{3}$$

The terms in ν take into account the damping due to the (turbulent) kinematic viscosity ν of the air, which will be assumed to be constant, and $(-fv, fu)$ is the Coriolis acceleration. The term fU, which plays here the part of a forcing term, takes into account the pressure gradient according to eq.(1).

(a) Show that

$$u = U(1 - e^{-z/\delta}\cos(z/\delta)) \ , \tag{4}$$

$$v = U(e^{-z/\delta}\sin(z/\delta)) \ , \tag{5}$$

assuming that the velocity of wind is $(u,v) = (0,0)$ at the ground level and $(u,v) = (U,0)$ at high altitude ($z \to \infty$).

(b) What is your physical interpretation for the length δ?

(c) Draw the trajectory of a small dead leaf (Lagrangian particle) that would be carried away by the local wind ((u,v) diagram).

(d) Taking $\delta = 500m$, compute the turbulent kinematic viscosity of the air.

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