MASTER [illegible]

1999 CAP University Prize Exam

Thursday, February 11

2:00 – 5:00 p.m.

Calculators are allowed.

Each question should be written in a different booklet, with the question number and your name and University clearly written on the first page.

You should attempt as many questions as possible, in whole or in part.

Questions are of equal value.

Send papers to:

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1. A wire of mass per unit length ρ vibrates in a half sine curve between two fixed points. An external circuit of resistance R is connected to the two fixed points forming a closed circuit with the vibrating wire. If a uniform magnetic field B is established normal to the plane of vibration, find the damping time for the amplitude to fall to $1/e$ of its initial value.

2. The door of an automobile stands fully open in a direction perpendicular to the longitudinal axis of the automobile. If the automobile, starting from rest, accelerates uniformly in the forward direction at a rate a, calculate the time required for the door to close. Assume that the door is a uniform rectangle of width w and treat the hinges as frictionless.

3. Suppose the following are measured on earth for a distant quasar. (i) the H_α absorption line, which has a wavelength $\lambda = 655.0$ nm on earth, is red-shifted to a wavelength $\lambda' = 982.5$ nm; (ii) it has a blackbody spectrum with a peak wavelength $\lambda'_m = 600.0$ nm; (iii) the energy from the quasar reaching earth per unit area per unit time is $R' = 10^{-13} W/m^2$.

   (a) What is the velocity v of the receding quasar with respect to us?

   (b) Assuming the velocity to be completely due to the expansion of the universe as a whole, what is the distance d of the quasar from us in Mpc?

   (c) What is the surface temperature T of the quasar, if a blackbody at 6000K is characterized by wavelength at the maximum of 489 nm. Remember to take into account the red shift here as well.

   (d) What is R (energy radiated per unit area per unit time) of the star, expressed in W/m^2? For this calculation you do not have to take into account possible relativistic corrections.

   (e) What is the radius r of the quasar in Mpc?

   Hint:

   Stefan-Boltzmann constant: $\sigma = 5.67 \times 10^{-8} W/m^2/K^4$
   Hubble constant: $H_0 = 70\ km/s/Mpc$
   1 Mpc = 3.26 light years.

4. A rectangular box of dimensions (a, b, c) with perfectly conducting walls is used as a resonator for electromagnetic waves.

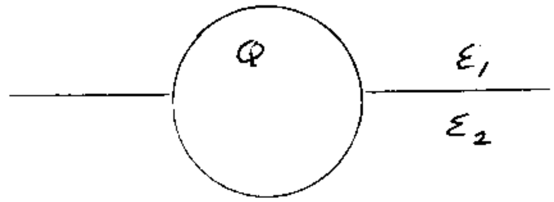
   (i) What are the boundary conditions on the components of $\vec{E}$ and $\vec{B}$ on the three pairs of plane surfaces?

   (ii) Find the components of $\vec{E}$ satisfying these conditions.

   (iii) The amplitudes of these three components are not all independent because the fields must satisfy Maxwell's Equations. What constraint is imposed by this condition?

   (iv) Determine the components of $\vec{B}$.

5. The equatorial plane of a spherical conductor lies in the plane interface between two semi-infinite dielectric media of permittivity ϵ_1 and ϵ_2, as illustrated in the accompanying figure. If the sphere is given a total free charge Q, find the surface charge density on the spherical conductor and on the contiguous dielectric surface. What is the total electrostatic energy of this system?



6. The Global Position System (GPS) consists of a number of satellites carrying extremely accurate atomic clocks, and relies on the precision measurement of tagged time signals. The satellites travel in circular orbits with periods of 12 hours.

The rate at which the clocks on the satellites run relative to those on earth will be affected by two relativistic effects. First, the satellite clocks run slightly slower due to the standard time dilation of special relativity. Secondly, according to general relativity, the satellite clocks run slightly faster due to their position in the earth's gravitational field. From laboratory tests, it is known that measurements with two atomic clocks can agree to one part in 10^{12}. To maximize the accuracy of the GPS, do the relativistic corrections have to be taken into account? Which effect is larger in the present case? To simplify your calculations, you may neglect the motion of the earth-bound clocks generated by the rotation of the earth.

The radius of the earth: $R_{\text{earth}} = 6,378\ km$

(**Hint:** For those less familiar with the gravitational effect, consider what dimensionless quantity must be relevant.)

7. The figure illustrates a soap film (shown shaded) supported by a wire frame. Because of surface tension the film exerts a force $2\sigma\ell$ on the cross wire ab. This force is in such a direction that it tends to move this wire so as to decrease the area of the film. The quantity σ is called the "surface tension" of the film and the factor 2 occurs because the film has two surfaces. The temperature dependence of σ is given by

$$\sigma = \sigma_0 - \alpha T$$

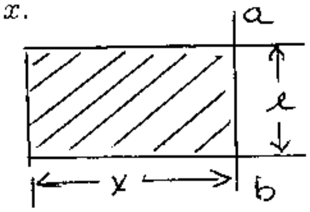
where σ_0 and α are constants independent of T and x.

(i) Suppose that the distance x (or equivalently, the total film area $2\ell x$) is the only external parameter of significance in the problem. Write a relation expressing the change dE in mean energy of the film in terms of the heat dQ absorbed by it and the work done by it in an infinitesimal quasi-static process in which the distance x is changed by an amount dx.

(ii) Calculate the change in mean energy $\Delta E = E(x) - E(0)$ of the film when it is stretched adiabatically from a length $x = 0$ to a length x.

Question #7, part (iii) continued on next page....

(iii) Calculate the work $W(0 \to x)$ done on the film in order to stretch it adiabatically from a length $x = 0$ to a length x.



8. Non-interacting electrons are moving freely in a two-dimensional plane (the x-y plane) which is inside a uniform magnetic field B pointing to the z-direction. The plane area is $A = L^2$ which is very large.

   (i) Derive the energy levels of an electron.

   (ii) Prove that the degeneracy of these levels per unit area is $N_0/A = 2eB/h$. Here e is the electron charge and h is Planck's constant.

   (iii) At very low temperatures ($k_B T \ll \hbar\omega_c$ where $\omega_c \equiv eB/m$), derive the magnetic field dependence of the chemical potential μ (the energy needed to add an electron to the system). Make a qualitative graphical display of $\mu(B)$.

9. The magnetic moment of a spin 1/2 particle is subjected to an inhomogeneous magnetic field with (x, y, z) components specified as,

$$(B\cos\omega t, B\sin\omega t, B_0)$$

   where B and B_0 are constant, independent of time and position. If at $t = 0$ at the spin was in the state $|0; +1/2>$, find the probability that at time t it will be found in the state $|t; -1/2>$.

10. A pencil (mass M, length R) is balanced vertically on its tip. Assume the tip doesn't slip as the pencil falls.

   (a) Suppose that the pencil starts at polar angle $\theta = \theta_0$ with initial angular momentum $L_x = L_\theta$, $L_y = L_z = 0$. Find an integral expression for the time it takes to reach an angle θ_f. Then find the leading behavior of this integral in the limit of small θ_0 and L_θ.

   (b) Suppose that initially the wave function for the angular orientation of the pencil is given by $\psi = Ne^{-a\theta^2}$, where N is the normalization constant and $a \gg 1$ so that the pencil is vertical to a good approximation. Find the uncertainty in θ and in the angular momentum $|\vec{L}|$ for this wave function.

   (c) Using the results of (a) and (b), estimate the value of a for which the pencil remains balanced as long as possible. Then estimate the amount of time an optimally balanced pencil of mass 100 g and length 10 cm takes to fall.

   The following integral may be useful:

$$\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}} \ln(2\sqrt{c(a + bx + cx^2)} + 2cx + b) \text{ for } \quad c > 0.$$