1997 CAP Undergraduate Prize Examination

Wednesday, February 5

from 2:00 to 5:00

Calculators are permitted.

Each question is to be written in a separate book with the number of the problem, the name of the candidate, and the name of the university indicated clearly on the first page.

The candidates may attempt as many questions as possible in whole or in part.

Each question holds the same value.

Please return the exams to:

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CAP Undergraduate Prize Exam

1 Capacitors with dielectrics

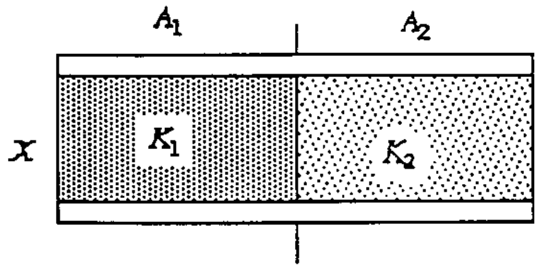


Fig. 1

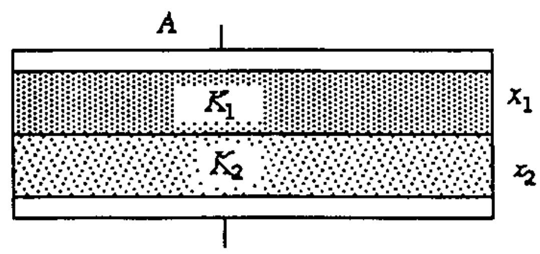


Fig. 2

In Fig. 1, A_1 is the area of the dielectric with dielectric constant K_1, A_2 is the area of the dielectric K_2, and x is the thickness of the dielectrics. In Fig. 2, A is the area of the dielectrics, x_1 is the thickness of the dielectric K_1, and x_2 is the thickness of the dielectric K_2.

(a) Find the capacitance for the arrangement shown in Fig. 1.

(b) Find the capacitance for the arrangement shown in Fig. 2.

(c) If a charge Q is applied to the capacitor shown in Fig. 1, what is the resulting energy densities in the regions with dielectric constant K_1 and K_2, respectively?

(Take $x_1 + x_2 = x$, and $A_1 + A_2 = A$.)

2. Interface plasmons.

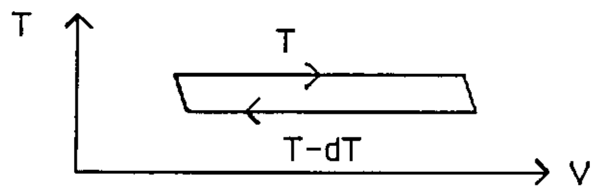
We consider the plane $z = 0$ between metal 1 with $z > 0$ and metal 2 with $z < 0$. Metal 1 has bulk plasma frequency ω_{p1}; Metal 2 has bulk plasma frequency ω_{p2}. A solution of Laplace's equation $\nabla^2\phi = 0$ in the plasma is $\phi_1(x,z) = A\cos kx e^{-kz}$ for $z > 0$ and $\phi_2(x,z) = A\cos kx e^{kz}$ for $z < 0$.

(a) Calculate E_x and E_z on each side of the boundary.

(b) Show that E_x is continuous across the boundary.

(c) Next, from the continuity of the z component of **D** at the boundary, and the fact that $\varepsilon_i(\omega) = 1 - \frac{\omega_{pi}^2}{\omega^2}$, for i =1 and 2, show that $\omega = \sqrt{\left(\omega_{p1}^2 + \omega_{p2}^2\right)/2}$.

3. In this problem, assume it is known that the energy density u (in J/m^3) for blackbody radiation is a function of the temperature T only, and also that the pressure $p = u/3$. The problem is to determine how u depends on T. This can be done as follows. Let the radiant energy in a cylinder be carried through a Carnot cycle, as shown in the diagram, consisting of an isothermal expansion at temperature T, an **infinitesimal** adiabatic expansion in which the temperature drops to T - dT, an isothermal compression at T - dT, and an **infinitesimal** adiabatic compression to the original state.



(a) Plot the cycle in the p - V plane.
(b) Calculate the work done by the system during the cycle.
(c) Calculate the heat flowing into the system during the cycle.
(d) Show that the energy density u is proportional to T^4 by considering the efficiency of the cycle.

4. The magnetic moment of an ion of spin J can have (2J+1) orientations with respect to an external field **B**. The components of the moment along **B** can be Jm, (J-1)m, (J-2)m,....(-J+1)m, -Jm. Consider a paramagnetic system of N *distinguishable* lattice sites, each occupied by one ion of spin J. (The ions are identical, except for being "nailed down", one to each site.) The paramagnetic system is in equilibrium at temperature $\tau = kT$, where k is the Boltzmann constant. Show that the magnetic moment M of the system is given by

$$M = Nm\left\{\left(J+\frac{1}{2}\right)\coth\left[\left(J+\frac{1}{2}\right)\frac{mB}{\tau}\right]-\frac{1}{2}\coth\left[\frac{1}{2}\frac{mB}{\tau}\right]\right\}$$

5. A point mass m_1 rests alone in space, while a distant second point mass m_2 moves with constant small velocity v_0. In the absence of gravity, m_2 would pass by m_1 with impact parameter (distance of closest approach) b_0. Taking gravity into account,

a) Find the energy and the angular momentum in the center of mass frame.

b) Find the actual impact parameter b in terms of b_0, v_0, G, and the two masses.

6. A rare mode of inverse β -decay involves resonant capture of an electron antineutrino $\bar{\nu}_e$ (assumed massless) by a hydrogen atom, producing only a recoil neutron. If the hydrogen atom is in its ground state and at rest, what would be the speed of the recoiling neutron?
$(m_H = 1.007825\,u,\ m_n = 1.008665\,u,\ 1\,u = 931.502\,MeV)$

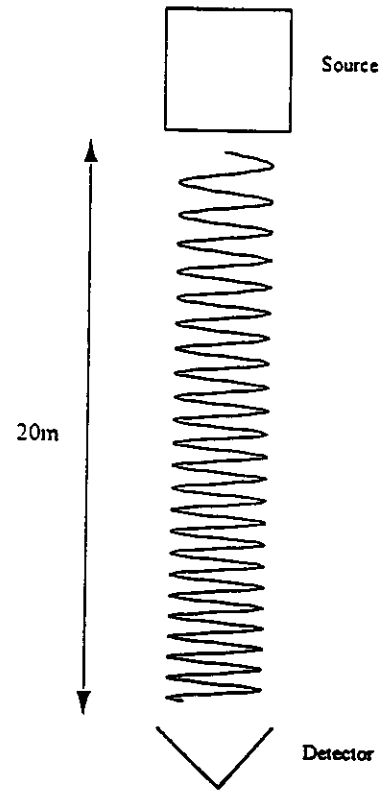
7. The existence of neutrino masses, and of oscillations between the three generations of neutrinos, remains an intriguing possibility. A toy model, which exhibits vacuum oscillations between two mass eigenstates, has the simple mass Hamiltonian

$$\mathbf{H} = \begin{pmatrix} M & m \\ m & M \end{pmatrix} c^2, \text{ where } M \geq m,$$

and the two physical states represented by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linear combinations of the mass eigenstates.

a) Find the eigenvalues of $\mathbf{H}$, and the corresponding mass eigenvectors.
b) By solving the time dependent Schrödinger equation
$i\hbar\partial_t \Psi(t) = \mathbf{H}\Psi(t)$,
find the shortest time τ necessary for a system, initially in the state represented by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, to transform completely into the state represented by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
c) If $Mc^2 = 17\,keV$, what would be the minimum value of τ in seconds?

8. One example of the Mossbauer effect is given by the decay of the ^{57}Co nucleus to ^{57}Fe by electron capture where the final nucleus (of mass M) subsequently returns to the ground state by emitting a photon. The energy of that photon depends on the total energy available, 14.4 keV in this case and labelled as E_∞, and upon the division of energy between the photon and the recoiling nucleus. The heavier the nucleus, the less energy is required to satisfy momentum conservation, and the smaller is the possible spread of energies of the emitted photon. If the nucleus is part of a macroscopic crystal then the recoiling mass can be very large and consequently the photon energy approaches that of a monoenergetic, monochromatic source - hence the label E_∞, the energy a photon would have if recoiling from an infinitely large mass.



a) Considering the emission of a photon from one single ^{57}Fe nucleus, use non-relativistic arguments (justified as the velocity of the recoil nucleus is small) applied to energy and momentum conservation to calculate the difference between E_∞ and the actual energy E_γ of the photon and show that the following aproximate relation holds:

$$E_\infty - E_\gamma \approx (E_\infty)^2/(2Mc^2)$$

b) Calculate the numerical value of the resulting frequency shift in the case of a photon recoiling against a large crystal of mass 1g.

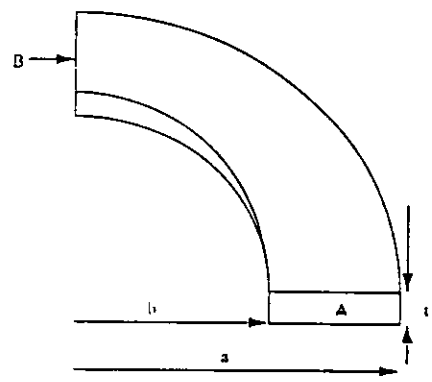
c) Using this as a source of monoenergetic, monochromatic X rays, what is the expected frequency shift when the photon falls 20m under gravity? Take the original frequency as 3.48×10^{18} Hz.

Constants:

$e = 1.6 \times 10^{-19}$ C
$c = 3 \times 10^{8}$ m/s
$h = 6.63 \times 10^{-34}$ J.s
$g = 9.8$ m/s^2

9. a) Material with a uniform resistivity ρ is formed into the curved shape shown below. The two curved surfaces are circular with radii of a and b and the thickness of the slab is uniform and equal to t . Find an expression for the electrical resistance between the two faces of the slab labelled A and B.

b) If the material is aluminum with conductivity $0.355 \times 10^{8}\ (\Omega.m)^{-1}$ and the object is machined such that a = 20 cm, b = 10cm and t = 1 cm, calculate numerically the resistance between those same two surfaces.



10. Explain briefly, in one paragraph per topic, the physics principles involved in the operation of any FOUR of the following devices:

i) a laser

ii) an Electric Field Meter

iii) a MOSFET transistor

iv) a proportional counter (for measuring radiation fields)

v) a thermocouple